Elliptical Gear Math

objective: Generate an expression that determines the rotation angle of an elliptical gear (β) knowing the rotation angle (θ), major radius (r_1), and minor radius (r_2) of its identical mating gear.

The two identical elliptical gears must each rotate about one of their respective foci. If lines are drawn from focus to focus as the elliptical gears rotate, mirrored triangles are formed.

Known:

- θ : angle of the driving gear.
- r_1: major radius
- r_2: minor radius

Discerned:

- a: short side of the formed triangle
- b: long side of the formed triangle
- c: distance between each gear's focus. c = 2f
- d: a constant, the sum of a and b. d = a+b

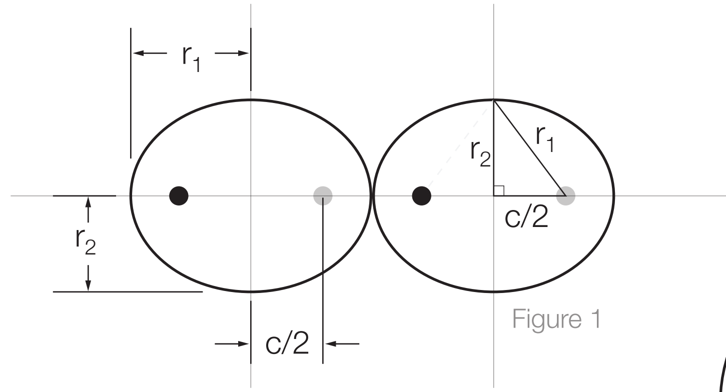

Figure 1

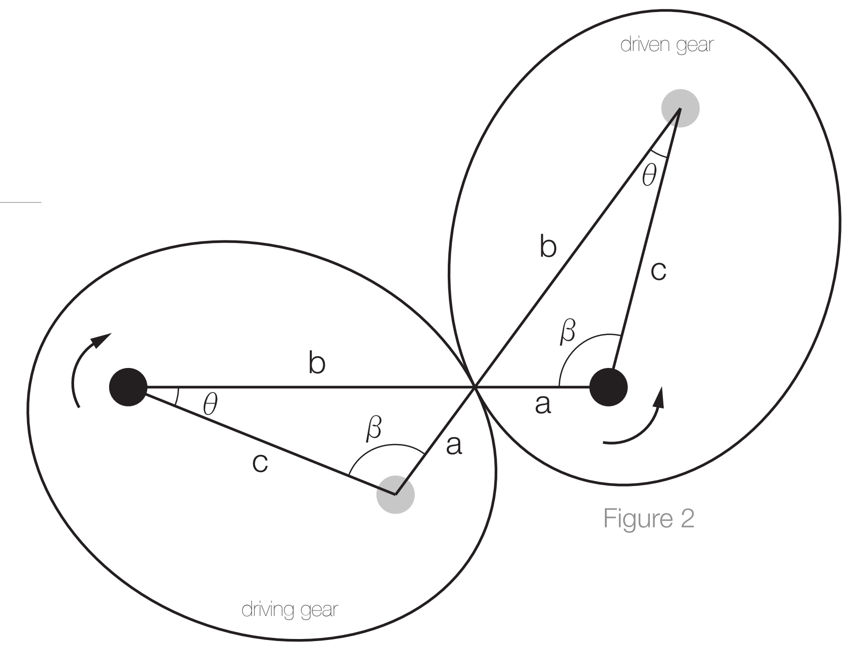

Figure 2

Since the two triangles are identical, the three known elements (c, θ , and d) on the driving gear's triangle can be used to calculate β on the driven gear's triangle, and thus, the rotation angle of the driven gear.

Clarification:

Why does d = a+b?

At any point along an ellipse, the sum of the line lengths formed by connecting that point to the two foci is constant.

This is easiest to visualize by building a simple model: Push two pins or thumb tacks into a board. These pins are the foci. Now, tie a string into a loop. Place that loop over the pins. The string is the triangle in Figure 2. If a pencil is placed in the loop, pulling it taught, the ellipse can be traced. Since the string's length does not change, and since the pin's distance from each other, line c, does not change, neither can the sum of a and b.

Why is r_1 the hypotenuse in Figure 1?

Look at Figure 2. The sum of lengths of lines a and b remains constant, d. Where θ reduces down to zero such that b passes though both foci and becomes oriented the same as Figure 1, it can be seen that d is equal to the major diameter of the ellipse, the length across the ellipse through both foci. As d is the major diameter, half of d is the major radius, r_1. So, $r_1 = d/2$. Imagine in Figure 1 that the triangle's hypotenuse is line a. If another line connected to the other focus, a and that line, b, would be equal. When a and b are equal, a is half of d, or $a = d/2 = r_1$, so r_1 is the hypotenuse.

First, β can be solved for in terms of c, d, and θ .

Law of cosines:

$$a^2 = b^2 + c^2 - 2bc\cos\theta$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$d = a + b$$

$$a = d - b$$

$$b = d - a$$

Substitute for b.

$$a^2 = (d-a)^2 + c^2 - 2(d-a)c\cos\theta$$

Then solve for a.

$$a = \frac{2cd\cos\theta - c^2 - d^2}{2(c\cos\theta - d)}$$

Substitute for a.

$$(d-b)^2 = b^2 + c^2 - 2bc\cos\theta$$

Then solve for b.

$$b = \frac{c^2 - d^2}{2(c\cos\theta - d)}$$

Solve for β .

$$\beta = \cos^{-1}\left(\frac{a^2 - b^2 + c^2}{2ac}\right)$$

Then substitute for a and b.

$$\beta = \cos^{-1}\left(\frac{\left(\frac{2cd\cos\theta - c^2 - d^2}{2(c\cos\theta - d)}\right)^2 - \left(\frac{c^2 - d^2}{2(c\cos\theta) - d)}\right)^2 + c^2}{2c\left(\frac{2cd\cos\theta - c^2 - d^2}{2(c\cos\theta - d)}\right)}\right)$$

Next, c and d can be solved for in terms of r_1 and r_2.

Pythagorean theorem:

$$r_1^2 = (c/2)^2 + r_2^2 \rightarrow c = 2\sqrt{r_1^2 - r_2^2}$$

$$r_1 = d/2 \rightarrow d = 2r_1$$

Finally, β can be solved for in terms of r_1 and r_2.

$$\beta = \cos^{-1}\left(\frac{\left(\frac{2\left(2\sqrt{r_1^2 - r_2^2}\right)(2r_1)\cos\theta - \left(2\sqrt{r_1^2 - r_2^2}\right)^2 - (2r_1)^2}{2\left(\left(2\sqrt{r_1^2 - r_2^2}\right)\cos\theta - (2r_1)\right)}\right)^2 - \left(\frac{\left(2\sqrt{r_1^2 - r_2^2}\right)^2 - (2r_1)^2}{2\left(\left(2\sqrt{r_1^2 - r_2^2}\right)\cos\theta - (2r_1)\right)}\right)^2 + \left(2\sqrt{r_1^2 - r_2^2}\right)^2}{2\left(2\sqrt{r_1^2 - r_2^2}\right)\left(\frac{2\left(2\sqrt{r_1^2 - r_2^2}\right)(2r_1)\cos\theta - \left(2\sqrt{r_1^2 - r_2^2}\right)^2 - (2r_1)^2}{2\left(\left(2\sqrt{r_1^2 - r_2^2}\right)\cos\theta - (2r_1)\right)}\right)}\right)$$